correctly that the negative experimental results could have been anticipated, had their stability analysis been performed in advance. The analysis showed that the effect of localized heating concentrated at the nose is different from the more common effect of heat transfer that is distributed over a large area, and hence different transition behavior can be expected.

The negative experimental results also could have been anticipated from the results of previous investigations. In 1942, Frick and McCullough[2] measured the variation in transition Reynolds number due to localized heating of various portions of a low-drag airfoil. They found that heating the upper surface of the airfoil caused an inflection point in the velocity profile, as expected, and a subsequent reduction in transition Reynolds number. However, when only the nose was heated, corresponding to Cebeci and Smith's case, there was no inflection point in the boundary layer far downstream and no change in transition Reynolds number.

More recently, McCroskey and Lam[3] found that heat applied at the leading edge of a flat plate produces no inflection point in the theoretical velocity profiles downstream, if the wall is adiabatic downstream. This result contrasts importantly with the well-known result that more uniform heating along a surface produces an inflection point and destabilizes the laminar boundary layer. In the experimental part of their investigation, McCroskey and Lam found that the isolated heat source at the origin actually delayed transition rather than promoting the onset of turbulence.

On the basis of the aforementioned results, it is, therefore, not surprising that Cebeci and Smith did not obtain earlier transition by heating the leading edge.

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Reply by Authors to W. J. McCroskey

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THE authors read Mr. McCroskey's comments with a mixture of pleasure and chagrin. The pleasure comes from the general agreement between their tests and the NACA Ames tests[1] cited by McCroskey. The chagrin comes from not finding Ref. 1. In researching the literature prior to the said experiment, the authors encountered Ref. 1; upon requesting this reference from their library, Ref. 2 was promptly obtained, a reference which has exactly the same report number, the same issue date, and the same authors as Ref. 1! Two thorough searches, by their library, through the entire Los Angeles area for Ref. 1 proved futile. The NACA Ames library verified that indeed the two reports were issued with the same number (Ref. 2 has subsequently been reissued as NACA TR 830). Thus, after a lengthy period of unsuccessful attempts to obtain Ref. 1, the authors gave up.

A few comments are in order about the tests of Ref. 1. These tests appear to be good and contain data that should be valuable to fundamental studies of the transition process. It does not deserve to be buried as an NACA Wartime Report. In this report, temperature differences of only 100°F were used. The authors tried much higher temperature differences, over 400°F, but the results were still negative. However, they did succeed in relating the experimental results to theory much more completely than in Ref. 1.

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Comment on "Strength Margins for Combined Random Stresses"

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Introduction

IN Ref. 1, the problem of finding the average frequency of crossing of a curve in the (x,y) plane by a Gaussian stationary random vector $\mathbf{z}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is considered. Two of the correlation coefficients defining the process were assumed to be zero. We will include all correlation coefficients and examine the problem where the curve can be conveniently approximated by straight line segments.

Average Crossing Frequency of a Straight Line Segment

Consider a straight line segment in the (x,y) plane from (x_1,y_1) to (x_2,y_2). Let (s,n) be orthogonal coordinates parallel to and normal to the segment respectively, with the same origin 0 as (x,y). In the (s,n) coordinates the segment lies between (s_1,n_1) and (s_2,n_1). Using the same procedure as in Ref. 1, we find the average crossing frequency per unit length of segment in the positive n direction $(\dot{n} > 0)$,

$$\bar{N}_{cp}(s,n) = \int_0^\infty \int_{-\infty}^\infty p(s,n,\dot{s},\dot{n})d\dot{s}\dot{n}d\dot{n} \tag{1}$$

where $p(s,n,\dot{s},\dot{n})$ is the probability density function of position and velocity of the vector $\mathbf{z}(t)$ in the (s,n) coordinates and $(\dot{\ })$ represents differentiation with respect to time. The corresponding average for crossings in the negative n direction $(\dot{n} < 0)$ is

$$\bar{N}_{cn}(s,n) = \int_{-\infty}^0 \int_{-\infty}^\infty p(s,n,\dot{s},\dot{n})d\dot{s}(-\dot{n})d\dot{n} \tag{2}$$

which, in general, is not the same as $\bar{N}_{cp}(s,n)$.

We now integrate to obtain the average frequency of crossing of the line segment in the positive n direction, say

$$N_{cp}(s_1,s_2,n_1) = \int_{s_1}^{s_2} \bar{N}_{cp}(s,n_1)ds \tag{3}$$

This compares with Eq. (11) of Ref. 1, where crossings in both directions are considered together, and a general curve C is taken instead of a straight line segment. In Ref. 1,

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$|\mathbf{N}\cdot\dot{\mathbf{z}}|$ is the velocity normal to the curve at the point under consideration.

Probability Density Function[2]

Let the column vector $\mathbf{x} = [xy\dot{x}\dot{y}]^T$ be the Gaussian stationary random process with zero expectation, $E(\mathbf{x}) = [0\ 0\ 0\ 0]^T$. The probability density function of the process is

$$p(\mathbf{x}) = k \exp\{-\tfrac{1}{2}(\mathbf{x}^T M^{-1}\mathbf{x})\}$$

where

$$k = (2\pi)^{-2}|M|^{-1/2}$$

and the matrix of covariances is

$$M = E[\mathbf{x}\cdot\mathbf{x}^T] = \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y & 0 & \rho_{x\dot{y}}\sigma_x\sigma_{\dot{y}} \\ & \sigma_y^2 & \rho_{y\dot{x}}\sigma_y\sigma_{\dot{x}} & 0 \\ & & \sigma_{\dot{x}}^2 & \rho_{\dot{x}\dot{y}}\sigma_{\dot{x}}\sigma_{\dot{y}} \\ \text{symmetrical} & & & \sigma_{\dot{y}}^2 \end{bmatrix} \quad (4)$$

Let $\mathbf{y} = [sn\dot{s}\dot{n}]^T$ be the transformed process with transformation matrix A such that

$$\mathbf{y} = A\mathbf{x}$$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad (5)$$

where θ is the angle $x0s$.

The transformed probability density function is

$$p(\mathbf{y}) = k \exp\{-\tfrac{1}{2}[\mathbf{y}^T(A^{-1})^T M^{-1} A^{-1}\mathbf{y}]\}|J_n| \quad (6)$$

where $J_n = |A^{-1}|$ is the Jacobian of the transformation. The diagonal and zero terms of M are as in Ref. 1, and the off-diagonal terms are found as follows:

$$\rho_{xy}\sigma_x\sigma_y = \int_0^\infty C_{xy}(\omega)d\omega$$

$$\rho_{\dot{x}\dot{y}}\sigma_{\dot{x}}\sigma_{\dot{y}} = \int_0^\infty C_{xy}(\omega)\omega^2 d\omega \quad (7)$$

$$\rho_{x\dot{y}}\sigma_x\sigma_{\dot{y}} = \int_0^\infty Q_{xy}(\omega)\omega dw = -\rho_{y\dot{x}}\sigma_y\sigma_{\dot{x}}$$

where $\Phi_{xy}(\omega) = C_{xy}(\omega) + iQ_{xy}(\omega)$ is the cross-power spectral density function of $x(t)$ and $y(t)$.

For convenience let

$$(A^{-1})^T M^{-1} A^{-1} = \begin{bmatrix} a & b & p & q \\ & c & r & t \\ & & d & e \\ \text{symmetrical} & & & f \end{bmatrix} \quad (8)$$

Then we can evaluate the integrals in Eqs. (1) and (2):

$$\bar{N}_{cp}(s,n) = \bar{N}L_p \quad (9)$$

$$\bar{N}_{cn}(s,n) = \bar{N}L_n \quad (10)$$

where

$$\bar{N} = (K/F)(2\pi/d)^{1/2}\exp\{-\tfrac{1}{2}(As^2 + 2Bsn + Cn^2)\}$$

$$L_p = 1 + \pi^{1/2}H\exp\{H^2\}(\text{erf}\{H\} - 1)$$

$$L_n = 1 + \pi^{1/2}H\exp\{H^2\}(\text{erf}\{H\} + 1)$$

and

$$A = a - p^2/d,\ B = b - rp/d,\ C = c - r^2/d$$

$$F = f - e^2/d,\ Q = q - ep/d,\ T = t - er/d \quad (11)$$

$$H = (nT + sQ)(2F)^{-1/2}$$

$\bar{N}_{cp}$ and $\bar{N}_{cn}$ are equal only when $H = 0$, in which case $L_p = L_n = 1$.

The integral over the segment has not been evaluated in the general case; however, it is well behaved for numerical integration using a polynomial approximation (e.g., Simpson's Rule). Two special cases with an analytical solution are now given.

I. Average crossing frequency in the positive n direction of the infinite straight line, $s_1 = -\infty$ to $s_2 = +\infty$ at $n = n_1$

$$N_{cp}(-\infty,\infty,n_1) = \frac{K}{F}\frac{2\pi}{(dA)^{1/2}}\exp\left\{-\tfrac{1}{2}n_1^2\left(C - \frac{B^2}{A}\right)\right\} + \frac{K}{F}\frac{2\pi}{(d\alpha)^{1/2}}\exp\left\{-\tfrac{1}{2}n_1^2\left(\gamma - \frac{\beta^2}{\alpha}\right)\right\} \times \left[\left\{\left(\frac{A}{\alpha}\right)^{1/2} - \left(\frac{\alpha}{A}\right)^{1/2}\right\}\exp\{-\mu^2\} + \pi^{1/2}\tau(\text{erf}\{\mu\} - 1)\right] \quad (12)$$

where

$$\alpha = A - Q^2/F,\ \beta = B - QT/F,\ \gamma = C - T^2/F$$

$$\tau = (n_1 T - Q\beta/\alpha)(2F)^{-1/2},\ \mu = \tau(\alpha/A)^{1/2}$$

This is the generalization of Rice's expression[1] for the average crossing frequency of a level $n = n_1$ of a two-dimensional process.

II. $\rho_{x\dot{y}} = \rho_{y\dot{x}} = 0$

In this case the line integral becomes uncoupled from the double velocity integral and

$$p = q = r = t = Q = T = H = 0 \text{ (zero)}$$

$$A = a, \quad B = b, \quad C = c$$

which yields

$$\bar{N}_{cp}(s,n) = \bar{N}_{cn}(s,n) = (K/F)(2\pi/d)^{1/2}\exp\{-\tfrac{1}{2}(as^2 + 2bsn + cn^2)\} \quad (13)$$

This should be compared with Eq. (22c) of Ref. 1, where $\rho_{x\dot{y}}$ is also zero, and the line integral has been indicated.

Integrating over the segment, we have

$$N_{cp}(s_1,s_2,n) = (K/F)[\pi/(ad)^{1/2}]\exp\{-\tfrac{1}{2}n_1^2(c - b^2/a)\}(\text{erf}\{C_2\} - \text{erf}\{C_1\}) \quad (14)$$

where

$$C_1 = (as_1 + b)(2a)^{-1/2},\ C_2 = (as_2 + b)(2a)^{-1/2}$$

and for the infinite line at n_1,

$$N_{cp}(-\infty,\infty,n) = (K/F)[2\pi/(ad)^{1/2}]\exp\{-\tfrac{1}{2}n_1^2(c - b^2/a)\} \quad (15)$$

Note that negative and positive crossings are the same in this case.

Discussion

For the same numerical example as in Ref. 1, we can examine the effect of the three correlation coefficients. Table 1 shows the average crossing frequency (in one direction) of the whole strength envelope per foot traveled, with an rms gust velocity of 50 fps.

As Fuller[1] notes, ρ_{xy} is the most important correlation coefficient. However, $\rho_{\dot{x}\dot{y}}$ has an appreciable effect on the

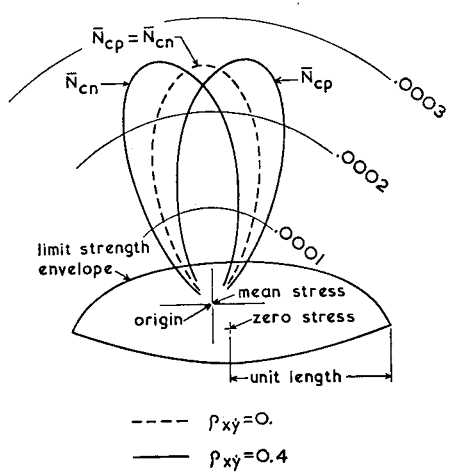

Fig. 1 Average crossing frequency of limit strength envelope per unit length of segment per ft traveled; rms gust velocity 50 fps.

average crossing frequency and should not be neglected, since it is easy to include in the exact analysis (see Eq. 14). The correlation coefficient $\rho_{x\dot{y}}$ has a negligible effect on the average crossing frequency of the whole strength envelope, but local effects are interesting.

Figure 1 shows a polar plot of the average crossing frequency per unit length of segment per foot traveled. (Aircraft velocity is assumed to be 650 fps.) Correlation coefficients ρ_{xy} and $\rho_{\dot{x}\dot{y}}$ are both zero and rms gust velocity is 50 fps. Superimposed on this is a diagram of the strength envelope scaled so that in the notation of Ref. 1, $\bar{A}_f = \bar{A}_\xi$, and having the origin at the mean stress point. The applied and allowable compressive stresses have been adjusted to the same effective area as the tension values, thus allowing the

Table 1 Average crossing frequency of limit strength envelope per ft traveled; rms gust velocity 50 fps

Case	ρ_{xy}	$\rho_{\dot{x}\dot{y}}$	$\rho_{x\dot{y}}$	$N_{cp}(= N_{cn}) \times 10^5$, per foot traveled	N_{cp}/N_{cp} (Case I)
I	0.	0.	0.	6.084	1.
II	0.4	0.	0.	5.866	0.964
III	0.	0.4	0.	5.944	0.977
IV	0.	−0.4	0.	6.216	1.022
V	0.	0.	±0.4	6.050	0.994

use of the tension mean and rms values for all direct stresses. Points on the strength envelope were determined by equally spaced radial lines from the origin. The straight line segments for crossing-frequency calculation join adjacent points on the envelope. The angular spacing used for this example was 0.1 rad, giving a minimum segment size near 0.025 unit length. Small perturbations of the strength envelope cause an appreciable change in the average crossing frequency. Most crossings occur in the tension region of the strength envelope, but for $\rho_{x\dot{y}} > 0$, there is a shift to the right of the outward (positive) crossing lobe, and a shift to the left of the inward (negative) crossing lobe. These lobes coincide for $\rho_{x\dot{y}} = 0$. It is conceivable that such behavior might affect the mode of fatigue failure under combined stress. Crossings in the compression region are two orders of magnitude less frequent than on the tension side.

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